

Any small multiplicative sugroup is not a sumset ^{*}

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Annotation.

We prove that for an arbitrary $\varepsilon > 0$ and any multiplicative subgroup $\Gamma \subseteq \mathbb{F}_p$, $1 \ll |\Gamma| \leq p^{2/3-\varepsilon}$ there are no sets $B, C \subseteq \mathbb{F}_p$ with $|B|, |C| > 1$ such that $\Gamma = B + C$. Also, we obtain that for $1 \ll |\Gamma| \leq p^{6/7-\varepsilon}$ and any $\xi \neq 0$ there is no a set B such that $\xi\Gamma + 1 = B/B$.

1 Introduction

Let $\mathbf{G} = (\mathbf{G}, +)$ be an abelian group and $B, C \subseteq \mathbf{G}$ be two sets. In Additive Combinatorics sets of the form $B + C := \{b + c : b \in B, c \in C\}$ are called the *sumsets* and studying properties of such sets is a central problem of this field, see [27]. A set A of integers is *additively reducible* if A cannot be written as a set of sums $B + C$ unless one of the sets consists of a single element, see [15], [17]. The question of additive irreducibility of integer sequences was posed by Ostmann [15] back in '56 (see also a modern overview [3]). Basically, Ostmann interested in reducibility of classical sets of Number Theory such as the primes numbers, shifted primes and so on. Sárközy [17] (see also [2], [10] and references therein) extended such problems to the finite field setting, perhaps for the first time suggesting that ‘multiplicatively structured’ sets should be additively irreducible. As a special case of his program, Sárközy conjectured that another classical object of Number Theory, namely, multiplicative subgroups of finite fields of prime order [12] are additively irreducible, in particular, the set of quadratic residues modulo a prime. Despite some progress (see, e.g., [20], [24] and references therein), the conjecture of Sárközy remains open and is considered as an important question in the field. The connected problems were covered in [4], [5], [9], [7], [13], [18] and in many other articles.

In papers [21], [22], [23] it was realized that this type of questions is connected with so-called the *sum-product phenomenon*, see [6], [27], [1], and the corresponding problem in the real setting was completely solved in [23], where it was proved, in particular,

Theorem 1 *There is $\varepsilon > 0$ such that for all sufficiently large finite $A \subset \mathbb{R}$ with $|AA| \leq |A|^{1+\varepsilon}$ there is no decomposition $A = B + C$ with $|B|, |C| > 1$.*

Here, of course, $AA = \{a_1 a_2 : a_1, a_2 \in A\}$ is the *product* set.

Let us formulate the main result of our paper.

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Theorem 2 *Let p be a prime number. Let $\varepsilon > 0$ be an arbitrary number, and Γ be a sufficiently large multiplicative subgroup, $|\Gamma| \geq C(\varepsilon)$, $|\Gamma| \leq p^{2/3-\varepsilon}$. Then for any $A, B \subseteq \mathbb{F}_p$, $|A|, |B| > 1$ one has $\Gamma \neq A + B$.*

Thus, we completely solved the problem on additive reducibility of multiplicative subgroups of size less than $p^{2/3}$.

In paper [18] Sárközy studied *multiplicative reducibility* of nonzero shifts of multiplicative subgroups Γ and proved that it is not possible to multiply arbitrary three sets to obtain such a shift (actually, Sárközy had deal with a particular case of quadratic residues). In our article we consider a symmetric situation and prove that $\xi\Gamma + 1 \neq B/B$ for any $\xi \neq 0$ and an arbitrary set B .

Theorem 3 *Let p be a prime number. Let $\varepsilon > 0$ be an arbitrary number, and Γ be a sufficiently large multiplicative subgroup, $|\Gamma| \geq C(\varepsilon)$, $|\Gamma| \leq p^{6/7-\varepsilon}$. Then for any $B \subseteq \mathbb{F}_p$, and an arbitrary $\xi \neq 0$ one has $\xi\Gamma + 1 \neq B/B$.*

The method of the proof develops the approach from [21]—[23] combining with [16], [14]. As it was realized in [21] that to prove $A \neq B + C$, $|B|, |C| > 1$ one must firstly separate our set A from the "random sumset case". Roughly speaking, it means that the most difficult case in the proof of $A \neq B + C$ is when $|B| \sim |C| \sim |A|^{1/2}$ and B, C look like random sets with the corresponding density. Further, one can note that if B, C are random sets, then $B + C$ is also behaves randomly more or less, so we must exploit some *nonrandom* properties of the sumset A . In [20] the author used the fact that if A is the set of quadratic residues, then A is so-called the *perfect difference set*, and the approach from [22], [23] is based on the *small doubling* property of A , namely, that $|AA| \ll |A|$. Of course a random set has such properties with probability zero. In this paper we continue to use latter non-random feature of A . More concretely, if $|AA| \ll |A|$ and, simultaneously, $A = B + C$, then A is both additively and multiplicatively rich and this contradicts with the sum-product phenomenon, see [6], [27], [1]. Unfortunately, at the moment the sum-product phenomenon over the prime finite fields is weaker then in the real setting, so we could not prove an analog of Theorem 1 in \mathbb{F}_p just copying the arguments of the proof of Theorem 1 (see the discussion section from article [23]). In our current approach we use some results from [16], [14] which are comparable with the correspondent theorems for the reals but on the other hand it requires to change the scheme of the prove from [23] somehow. So, we apply weaker incidence bounds than were used in [23] and, in particular, we reprove the results of this paper. Interestingly, that both methods of paper [23] and the current one use a nontrivial upper bound for the additive energy of multiplicative subgroups and multiplicatively rich sets, see Theorem 8 below.

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2 Definitions and preliminaries

The following notation is used throughout the paper. Let p be a prime number. By \mathbb{F}_p we denote the prime field. The expressions $X \gg Y$, $Y \ll X$, $Y = O(X)$, $X = \Omega(Y)$ all have the same

meaning that there is an absolute constant $c > 0$ such that $|Y| \leq c|X|$. All logarithms are to base 2.

For sets A and B from \mathbb{F}_p the *sumset* $A + B$ is the set of all pairwise sums

$$A + B = \{a + b : a \in A, b \in B\},$$

and similarly AB , $A - B$ denotes the set of *products* and *differences*, respectively. We denote by $|A|$ the cardinality of a set A . The *additive energy* $E^+(A)$, see [27], denotes the number of additive quadruples (a_1, a_2, a_3, a_4) such that $a_1 + a_2 = a_3 + a_4$. We use representation function notations like $r_{AB}(x)$ or $r_{A+B}(x)$, which counts the number of ways $x \in \mathbb{F}_p$ can be expressed as a product ab or a sum $a + b$ with $a \in A$, $b \in B$, correspondingly. For example, $|A| = r_{A-A}(0)$ and $E^+(A) = r_{A+A-A-A}(0) = \sum_x r_{A+A}^2(x) = \sum_x r_{A-A}^2(x)$.

Now we are ready to introduce the main object of our paper. Let $A, B, C, D \subseteq \mathbb{F}_p$ be four sets. By $Q(A, B, C, D)$ we denote the number of *collinear quadruples* in $A \times A$, $B \times B$, $C \times C$, $D \times D$. If $A = B = C = D$, then we write $Q(A)$ for $Q(A, A, A, A)$. Recent results on the quantity $Q(A)$ can be found in [16] and [14]. It is easy to see (or consult [14]) that

$$Q(A, B, C, D) = \left| \left\{ \frac{b' - a'}{b - a} = \frac{c' - a'}{c - a} = \frac{d' - a'}{d - a} : a, a' \in A, b, b' \in B, c, c' \in C, d, d' \in D \right\} \right| \quad (1)$$

$$= \sum_{a, a' \in A} \sum_x r_{(B-a)/(B-a')}(x) r_{(C-a)/(C-a')}(x) r_{(D-a)/(D-a')}(x). \quad (2)$$

Notice that in (1), we mean that the condition, say, $b = a$ implies $c = d = b = a$ or, in other words, that all four points (a, a') , (b, b') , (c, c') , (d, d') have the same abscissa. More rigorously, the summation in (2) should be taken over $\mathbb{F}_p \cup \{+\infty\}$, where $x = +\infty$ means that the denominator in any fraction $x = \frac{b'-a'}{b-a}$ from, say, $r_{(B-a)/(B-a')}(x)$ equals zero. Anyway, it is easy to see that the contribution of the point $+\infty$ is at most $O(M^5)$, where $M = \max\{|A|, |B|, |C|, |D|\}$, and hence it is negligible (see, say, Theorem 4 below). Further defining a function $q_{A,B,C,D}(x, y)$ (see [14]) as

$$q_{A,B,C,D}(x, y) := \left| \left\{ \frac{b-a}{c-a} = x, \frac{d-a}{c-a} = y : a \in A, b \in B, c \in C, d \in D \right\} \right|, \quad (3)$$

we obtain another formula for the quantity $Q(A, B, C, D)$, namely,

$$Q(A, B, C, D) = \sum_{x, y} q_{A,B,C,D}^2(x, y).$$

An optimal (up to logarithms factors) upper bound for $Q(A)$ was obtained in [14], [16].

Theorem 4 *Let $A \subseteq \mathbb{F}_p$ be a set. Then*

$$Q(A) = \frac{|A|^8}{p^2} + O(|A|^5 \log |A|).$$

In particular, if $|A| \leq p^{2/3}$, then $Q(A) \ll |A|^5 \log |A|$.

We need in a simple lemma about a generalization of the quantity $Q(A)$. The proof is analogous of the proof [23, Lemma 6].

Lemma 5 *Let $A, B \subseteq \mathbb{F}_p$ be two sets, $|B| \leq |A| \leq \sqrt{p}$. Then*

$$Q(A, B, A, B) \ll |A|^{5/2} |B|^{5/2} \log^2 |A| + |A|^3 |B|^2. \quad (4)$$

Proof. Write $L_{i,j}$, $i, j \geq 0$ for the set of lines ℓ such that $2^i \leq |\ell \cap (A \times A)| < 2^{i+1}$ and $2^j \leq |\ell \cap (B \times B)| < 2^{j+1}$. Then

$$Q(A, B, A, B) \ll \sum_{i=0}^{\log |A|} \sum_{j=0}^{\log |B|} |L_{i,j}| 2^{2i} 2^{2j}. \quad (5)$$

First of all, consider the lines ℓ with $j = 0$. In other words, each of such a line intersects $B \times B$ exactly at one point. Then we choose a point from A , forming a line intersecting $A \times A$, and obtain at most $|A|$ points on each of these lines. It gives us at most $|A|^3$ incidences. Another proof of this fact is the following. Fixing b, b' in (1), we have at most $O(|A|^3)$ possibilities for $a, a', c, c' \in A$ such that $\frac{b'-a'}{b-a} = \frac{c'-a'}{c-a}$. Totally, it gives at most $O(|B|^2 |A|^3)$ collinear triples. Clearly, the same arguments take place for $i = 0$, so we suppose below that $i, j \geq 1$.

Since the number of summands in (5) is at most $\log^2 |A|$ it is enough to bound each term by $|A|^{5/2} |B|^{5/2}$. For the sake of notation, denote $k = 2^i$ and $l = 2^j$, $L = L_{i,j}$, so that our task is to estimate $|L| k^2 l^2$ where L is the set of lines intersecting $A \times A$ in k (up to a factor of two) points and $B \times B$ in l points (again, up to a factor of two). Here $k \geq 2, l \geq 2$.

By the assumption $|B| \leq |A| \leq \sqrt{p}$. It implies that $2|A|^2/p \leq 2 \leq k, l$. The arguments of the proof of Theorem 4 gives us (see [14, Lemma 14]) that for $k, l \geq 2$ the following incidence estimate holds

$$|L| \ll \min \left(\frac{|A|^5}{k^4} + \frac{|A|^2}{k}, \frac{|B|^5}{l^4} + \frac{|B|^2}{l} \right),$$

so

$$T := k^2 l^2 |L| \ll \min \left(\frac{l^2 |A|^5}{k^2} + k l^2 |A|^2, \frac{k^2 |B|^5}{l^2} + k^2 l |B|^2 \right).$$

Since $k \leq |A|, l \leq |B|$, we see that the first terms dominate in the last formula. Hence multiplying, we get

$$T^2 \ll \frac{l^2 |A|^5}{k^2} \cdot \frac{k^2 |B|^5}{l^2} = |A|^5 |B|^5$$

as required. \square

Remark 6 *It is easy to see that we need in the term $|A|^3 |B|^2$ in (4). Indeed, take $B = \{0\}$ and A be a multiplicative subgroup in \mathbb{F}_p . Then one can check that $Q(A, B, A, B) = |A|^3$ but not $O(|A|^{5/2} \log^2 |A|)$.*

By $\mathsf{T}(A, B, C)$ denote the number of *collinear triples* in $A \times A$, $B \times B$, $C \times C$, where $A, B, C \subseteq \mathbb{F}_p$ are three sets. It is easy to check that

$$\mathsf{T}(A, B, C) = \left| \left\{ \frac{b-a}{c-a} = \frac{b'-a'}{c'-a'} : a, a' \in A, b, b' \in B, c, c' \in C \right\} \right| = \sum_x t_{A,B,C}^2(x),$$

where $t_{A,B,C}(x) = \sum_{a \in A} r_{(B-a)/(C-a)}(x)$. The support of the function $t_{A,B,C}(x)$ is denoted by $\mathsf{T}[A, B, C]$. One has (see [22])

$$\mathsf{T}[A, B, A] = 1 - \mathsf{T}[A, B, A]. \quad (6)$$

If $A = B = C$, then we write $\mathsf{T}[A]$ for $\mathsf{T}[A, B, C]$ and $\mathsf{T}(A)$ for $\mathsf{T}(A, B, C)$. Let us recall a result about $\mathsf{T}[A]$ from [14].

Theorem 7 *Let $A \subseteq \mathbb{F}_p$. Then*

- $|\mathsf{T}[A]| \gg \min \left\{ p, \frac{|A|^{5/2}}{p^{1/2}} \right\}.$
- $|\mathsf{T}[A]| \gg \min \left\{ p, |A|^{\frac{3}{2} + \frac{1}{22} - o(1)} \right\},$ with the $o(1)$ term tending to 0 as $p \rightarrow \infty$.
- $|\mathsf{T}[A]| \gg \min \left\{ p^{2/3}, \frac{|A|^{8/5}}{\log^{28/15} |A|} \right\}.$

Now let us remind some results about multiplicative subgroups. We need a simplified version of Theorem 8 from [19].

Theorem 8 *Let p be a prime number and $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup, $|\Gamma| \leq p^{2/3}$. Then*

$$\mathsf{E}^+(\Gamma) \ll |\Gamma|^3 p^{-\frac{1}{3}} \log |\Gamma| + p^{\frac{1}{26}} |\Gamma|^{\frac{31}{13}} \log^{\frac{8}{13}} |\Gamma|, \quad (7)$$

and

$$\mathsf{E}^+(\Gamma) \ll |\Gamma|^{\frac{32}{13}} \log^{\frac{41}{65}} |\Gamma|, \quad (8)$$

provided $|\Gamma| < p^{1/2} \log^{-1/5} p$.

Corollary 9 *Let $\varepsilon > 0$ be a positive real and $\Gamma \subseteq \mathbb{F}_p$ be a multiplicative subgroup, $|\Gamma| \leq p^{2/3-\varepsilon}$. Then for some $\delta(\varepsilon) > 0$ one has $\mathsf{E}^+(\Gamma) \ll |\Gamma|^{5/2-\delta(\varepsilon)}$.*

Because always $\mathsf{E}^+(\Gamma) \geq |\Gamma|^4/p$, it follows that $\mathsf{E}^+(\Gamma) \gg |\Gamma|^{5/2}$ for $|\Gamma| \sim p^{2/3}$. Thus, the constant $2/3$ in Corollary 9 is optimal.

The last needed result is the main theorem from [25].

Theorem 10 *Let $\Gamma \subseteq \mathbb{F}_p$ be a multiplicative subgroup, $k \geq 1$ be a positive integer, and x_1, \dots, x_k be different nonzero elements. Also, let*

$$32k2^{20k \log(k+1)} \leq |\Gamma|, \quad p \geq 4k|\Gamma|(|\Gamma|^{\frac{1}{2k+1}} + 1). \quad (9)$$

Then

$$|\Gamma \cap (\Gamma + x_1) \cap \dots \cap (\Gamma + x_k)| \leq 4(k+1)(|\Gamma|^{\frac{1}{2k+1}} + 1)^{k+1}. \quad (10)$$

Further

$$|\Gamma \cap (\Gamma + x_1) \cap \dots \cap (\Gamma + x_k)| = \frac{|\Gamma|^{k+1}}{(p-1)^k} + \theta k 2^{k+3} \sqrt{p}, \quad (11)$$

where $|\theta| \leq 1$. The same holds if one replaces Γ in (10) by any cosets of Γ .

Thus, the theorem above asserts that $|\Gamma \cap (\Gamma + x_1) \cap \dots (\Gamma + x_k)| \ll_k |\Gamma|^{\frac{1}{2} + \alpha_k}$, provided $1 \ll_k |\Gamma| \ll_k p^{1 - \beta_k}$, where α_k, β_k are some sequences of positive numbers, and $\alpha_k, \beta_k \rightarrow 0$, $k \rightarrow \infty$. A little bit better bounds than in (9), (10) can be found in [26].

3 The proof of Theorem 2

Let us formulate the main technical result of this section.

Proposition 11 *Let Γ be a multiplicative subgroup, and $A, B \subseteq \mathbb{F}_p$ be arbitrary sets such that $|B| \leq |A| \leq \sqrt{p}$ and for some nonzero η_1, η_2 and two sets $\Omega_1, \Omega_2 \subseteq \mathbb{F}_p$, $|\Omega_1|, |\Omega_2| \leq |\Gamma|$ the following holds*

$$\mathsf{T}[B, A, A] \subseteq \eta_1 \Gamma \cup \Omega_1 \quad \text{and} \quad \mathsf{T}[A, B, B] \subseteq \eta_2 \Gamma \cup \Omega_2. \quad (12)$$

Then

$$|A|^4 |B|^4 |\Gamma| \ll (\mathsf{E}^+(\Gamma) + \omega |\Gamma|^2 + |\Gamma|^2) \cdot \left((|A||B|)^{5/2} \log^2 |A| + |A|^3 |B|^2 \right), \quad (13)$$

where $\omega = \max\{|\Omega_1|, |\Omega_2|\}$.

Proof. Let $q(x, y) := q_{A, B, A, B}(x, y)$. Thus, any pair (x, y) from the support of the function $q(x, y)$ can be represented as $x = \frac{b-a}{a'-a}$, $y = \frac{b'-a}{a'-a}$, where $a, a' \in A$, $b, b' \in B$. It is easy to see that $\frac{1}{x} = \frac{a'-a}{b-a}$ and because (12)

$$1 - \frac{1}{x} = 1 - \frac{a' - a}{b - a} = \frac{b - a'}{b - a} \in \mathsf{T}[B, A, A] \subseteq \eta_1 \Gamma \cup \Omega_1.$$

Hence $x, y \in (1 - (\eta_1 \Gamma \cup \Omega_1))^{-1} \cup \{0\} := S_1$. Also, notice that in view of the second condition from (12), we have for $a \neq a'$ that

$$\frac{x}{y} = \frac{b - a}{b' - a} \in \mathsf{T}[A, B, B] \subseteq \eta_2 \Gamma \cup \Omega_2 := S_2.$$

These inclusions allow us to obtain a good upper bound for size of the support of the function $q(x, y)$. Let $\sigma := |\text{supp } q|$. By Lemma 5, we get

$$\begin{aligned} (|B|^2 |A|^2)^2 &\ll \left(\sum_{x, y} q(x, y) \right)^2 \leq \sum_{x, y} S_1(x) S_1(y) S_2(x/y) \cdot \sum_{x, y} q^2(x, y) = \sigma \cdot \mathsf{Q}(A, B, A, B) \quad (14) \\ &\ll \sigma \left((|A||B|)^{5/2} \log^2 |A| + |A|^3 |B|^2 \right). \end{aligned}$$

In the first inequality of (14) we have used the condition that $|A|, |B| > 1$. It remains to estimate the sum σ . Let σ' be the subsum of σ , where all variables $x, y, x/y$ do not belong to $(1 - \Omega_1)^{-1} \cup \Omega_2 \cup \{0\}$ and let σ'' be the rest. We have

$$\begin{aligned} \sigma' &\leq \left| \left\{ \frac{1 - \eta_1 \gamma_1}{1 - \eta_1 \gamma_2} = \eta_2 \gamma \quad : \quad \gamma_1, \gamma_2, \gamma \in \Gamma \right\} \right| = |\{1 - \eta_1 \gamma_1 = \eta_2 \gamma (1 - \eta_1 \gamma_2) \quad : \quad \gamma_1, \gamma_2, \gamma \in \Gamma\}| \quad (15) \\ &= |\{1 - \eta_1 \gamma_1 = \eta_2 \gamma - \eta_1 \eta_2 \gamma'_2 \quad : \quad \gamma_1, \gamma'_2, \gamma \in \Gamma\}| = \end{aligned}$$

$$= |\Gamma|^{-1} \mathbf{E}^+(\Gamma, -\eta_1 \Gamma, \eta_2 \Gamma, -\eta_1 \eta_2 \Gamma) \leq |\Gamma|^{-1} \mathbf{E}^+(\Gamma). \quad (16)$$

Here $\mathbf{E}^+(A, B, C, D) = |\{(a, b, c, d) \in A \times B \times C \times D : a + b = c + d\}|$ and the last bound in (16) is a consequence of the Hölder inequality. Finally, using a crude bound for the sum σ'' , we obtain

$$\sigma'' \leq 3\omega(|\Gamma| + \omega) + 3\omega(|\Gamma| + \omega) + 3(|\Gamma| + \omega) \leq 12\omega|\Gamma| + 6|\Gamma|.$$

This completes the proof. \square

Remark 12 *Using Theorem 10 with $k = 1$, one can improve the error term $\omega|\Gamma|^2 + |\Gamma|^2$ in (13). Indeed, for $|\Gamma| < p^{3/4}$, say, one can replace it by $\omega|\Gamma|^{5/3} + |\Gamma|^2$. It gives more room in inclusions (12) allowing ω be $|\Gamma|^{5/6-\varepsilon_0}$ for some $\varepsilon_0 > 0$, see Corollary 13 below.*

Now we are ready to prove Theorem 2 from the introduction.

Corollary 13 *Let $\varepsilon > 0$ be an arbitrary number, and Γ be a sufficiently large multiplicative subgroup, $|\Gamma| \geq C(\varepsilon)$, $|\Gamma| \leq p^{2/3-\varepsilon}$. Then for any $A, B \subseteq \mathbb{F}_p$, $|A|, |B| > 1$ one has $\Gamma \neq A + B$.*

Proof. Suppose that for some $A, B \subseteq \mathbb{F}_p$, $|A|, |B| > 1$ the following holds $\Gamma = A + B$. Without loss of generality suppose that $|B| \leq |A|$. Redefining $B = -B$, we see that conditions (12) take place with $\eta_1 = \eta_2 = 1$ and $\Omega_1 = \Omega_2 = \{0\}$. By [24], [25] (or just use Theorem 10 above) we know that $|A| \sim |B| \sim |\Gamma|^{1/2+o(1)}$. In particular, it gives us for Γ large enough that $|B| \leq |A| \leq \sqrt{p}$. Thus, applying Proposition 11, we obtain

$$|A|^8 |B|^8 |\Gamma|^2 \ll \mathbf{E}^+(\Gamma)^2 (|A|^5 |B|^5 + |A|^6 |B|^4) \log^4 |\Gamma| \ll |\Gamma|^{o(1)} \mathbf{E}^+(\Gamma)^2 |A|^5 |B|^5 \log^4 |\Gamma|.$$

By Corollary 9, we find some $\delta = \delta(\varepsilon) > 0$ such that $\mathbf{E}^+(\Gamma) \ll |\Gamma|^{5/2-\delta}$ and hence

$$|A|^3 |B|^3 \ll |\Gamma|^{3-2\delta+o(1)}.$$

Clearly, if $\Gamma = A + B$, then $|A||B| \geq |\Gamma|$. Whence

$$|\Gamma|^3 \ll |\Gamma|^{3-2\delta+o(1)}.$$

and it gives a contradiction for large Γ . This completes the proof. \square

Remark 14 *Actually, our results give an effective bound for sizes of sets A, B with $A + B \subseteq \Gamma$. It has the form $\min\{|A|, |B|\} \ll |\Gamma|^{1/2-c/2}$, where $c > 0$ is an absolute constant (for similar problems, see [9]).*

Remark 15 *It is easy to see that one can replace the condition $A - B \subseteq \Gamma$ onto $A - B \subseteq \Gamma \cup \{0\}$, say, or, generally speaking, onto $A - B \subseteq \Gamma \sqcup \Omega$, where $|\Omega| = O(|\Gamma|^{o(1)})$. Moreover, our arguments take place for sets A, B with $A - B \subseteq \bigsqcup_{j=1}^s \xi_j \Gamma$, where $s = O(|\Gamma|^{o(1)})$ and $\xi_j \Gamma$ some cosets of Γ .*

Notice that for small Γ it can be $\Gamma \sqcup \{0\} = A - B$, say. For example (see [13]), let $p = 13$, $\Gamma = \{1, 3, 4, 9, 10, 12\}$, $A = B = \{2, 5, 6\}$. Then one can check that, indeed, $A - A = \Gamma \sqcup \{0\}$. More generally, for any subgroup A , $|A| = 2$ or $|A| = 3$ one has $\Gamma \sqcup \{0\} = \xi(A - A)$ for some ξ and some subgroup Γ (see [21] or just use a direct calculation). Thus, the condition $|\Gamma| \gg 1$ is required in the corollary above.

The next corollary is connected with the main result from [25], see Theorem 10 above. Bound (17) below is better than (10) for very large k .

Corollary 16 *Let Γ be a multiplicative subgroup, $|\Gamma| < p^{1/2} \log^{-1/5} p$. Let also $x_1, \dots, x_k \in \mathbb{F}_p$ be any distinct numbers, $k \leq |\Gamma|^{\frac{19}{39}} \log^{-\frac{219}{195}} |\Gamma|$. Then*

$$|(\Gamma + x_1) \cap (\Gamma + x_2) \cap \dots \cap (\Gamma + x_k)| \ll k^{-2} |\Gamma|^{\frac{19}{13}} \log^{\frac{41}{65}} |\Gamma|. \quad (17)$$

Proof. Put $A = (\Gamma + x_1) \cap (\Gamma + x_2) \cap \dots \cap (\Gamma + x_k)$ and $B = \{x_1, \dots, x_k\}$. If estimate (17) takes place, then there is nothing to prove. Otherwise, we get $|A| \gg k^{-2} |\Gamma|^{\frac{19}{13}} \log^{\frac{41}{65}} |\Gamma|$. Further, we have $A - B \subseteq \Gamma$. Using Proposition 11 and our condition on k , namely, $k \leq |\Gamma|^{\frac{19}{39}} \log^{-\frac{219}{195}} |\Gamma|$, one has

$$|A|^4 k^4 |\Gamma| \ll E^+(\Gamma) (|A|^{5/2} k^{5/2} \log^2 |\Gamma| + |A|^3 k^2) \ll E^+(\Gamma) |A|^3 k^2.$$

Applying Theorem 8, we obtain the required result. \square

4 The proof of Theorem 3

It remains to prove our second result, i.e. Theorem 3. We begin with a lemma which is parallel to the main results of [18], [24] and we just repeat the proof from [24] (although in Theorem 19 below we need in the case $|\Gamma| \leq p^{6/7-\varepsilon}$ only).

Lemma 17 *Let $\Gamma \subset \mathbb{F}_p \setminus \{0\}$ be a multiplicative subgroup, $A, B \subseteq \mathbb{F}_p$ be two sets, $|A \setminus \{0\}| > 1$, $|B \setminus \{0\}| > 1$. Suppose that for some $x \neq 0$ one has $A/B = \Gamma + x$. Then $|A|, |B| \sim |\Gamma|^{1/2+o(1)}$.*

Proof. Dividing by x and redefining A , we have $A/B = \xi\Gamma + 1$, where $\xi = 1/x$. In other words, for any $a \in A$, $b \in B$, $b \neq 0$ one has

$$\frac{a}{b} - 1 = \frac{a-b}{b} \in \xi\Gamma. \quad (18)$$

One can assume that $1 \in B$ and hence $A \subseteq \xi\Gamma + 1$. Put $A' = A \setminus \{0\}$, $B' = B \setminus \{0\}$, and assume, in addition, that $|A| \geq |B|$. In view of (18) for any $b_1, \dots, b_k \in B'$ one has

$$A \subseteq (b_1\xi\Gamma + b_1) \cap (b_2\xi\Gamma + b_2) \cap \dots \cap (b_k\xi\Gamma + b_k). \quad (19)$$

Since $|B'| > 1$, it follows that for some different $b_1, b_2 \in B'$, we get $b_1^{-1}A - 1, b_2^{-1}A - 1 \subseteq \xi\Gamma$ and hence $b_1^{-1}b_2^{-1}(A - b_1)(A - b_2) \subseteq \xi^2\Gamma$. Fix $\varepsilon > 0$ and suppose, firstly, that $|\Gamma| \leq p^{1-\varepsilon}$. By a sum-product result from [8], [1], we see $|(A - b_1)(A - b_2)| \gg |A|^{1+\delta}$, where $\delta = \delta(\varepsilon) > 0$. It immediately implies $|A| \ll |\Gamma|^{1-\delta/2}$. Thus, because, trivially, $|A||B| \geq |\Gamma|$, we obtain $|B| \gg |\Gamma|^{\delta/2}$ and we can use inclusion (19) and Theorem 10 with $k = |\Gamma|^{\delta/2}$ to reach, finally, $|B| \leq |A| \ll |\Gamma|^{1/2+o(1)}$. Again, $|A||B| \geq |\Gamma|$ and hence $|A| \sim |B| \sim |\Gamma|^{1/2+o(1)}$.

Now let $|\Gamma| > p^{1-\varepsilon}$. Let \mathcal{X} be the set of all multiplicative characters χ such that $\chi^d = \chi_0$, where χ_0 is the principal character and $d = (p-1)/|\Gamma| < p^\varepsilon$. Also, put $\mathcal{X}^* = \mathcal{X} \setminus \{0\}$. If $|A| \geq |B| > p^\delta$ (actually, we have $|A| \geq |\Gamma|^{1/2}$ automatically, since $|A| \geq |B|$ and $|A||B| \geq |\Gamma|$)

for some positive δ , then taking any $\chi \in \mathcal{X}^*$, we obtain $|\sum_{a \in A'} \sum_{b \in B'} \chi(a-b) \overline{\chi(b)}| = |A'| |B'|$. Combining this with the Karatsuba bound [11]

$$\sum_{a \in A} \sum_{b \in B} \alpha(a) \beta(b) \chi(a+b) \ll |A|^{1-1/2\nu} \left(|B|^{1/2} p^{1/2\nu} + |B| p^{1/4\nu} \right)$$

which takes place for any positive integer ν and any sequences α, β such that $\|\alpha\|_\infty \leq 1, \|\beta\|_\infty \leq 1$, we obtain

$$|A| |B| \ll |A|^{1-1/2\nu} \left(|B|^{1/2} p^{1/2\nu} + |B| p^{1/4\nu} \right) \ll |A|^{1-1/2\nu} |B| p^{1/4\nu}.$$

Here we have taken $\nu = \lceil 1/\delta \rceil$ and have used $|A| \geq |B| > p^\delta$. Thus, $|A| \ll \sqrt{p} \ll |\Gamma|^{1/2+o(1)}$ as required.

To insure that $|A| \geq |B| > p^\delta$ we just notice that for any $u \in \xi\Gamma + 1$ there is $b \in B'$ such that $ub \in A \subseteq \xi\Gamma + 1$. In other words, $\xi^{-1}(ub-1) \in \Gamma$. We have $\Gamma(x) = \frac{1}{d} \sum_{\chi \in \mathcal{X}} \chi(x)$ and hence by the last inclusion, we get for $\{b_1, \dots, b_k\} = B' \setminus \{1\}$ that

$$\begin{aligned} 0 &= \sum_{u \in \xi\Gamma+1} \prod_{j=1}^k (1 - (\xi\Gamma+1)(ub_j)) = \sum_{u \in \xi\Gamma+1} \prod_{j=1}^k \left(1 - \frac{1}{d} \sum_{\chi \in \mathcal{X}} \chi(ub_j-1) \overline{\chi(\xi)} \right) = \\ &= \sum_{x \in \mathbb{F}_p \setminus \{0\}} \prod_{j=1}^k \left(1 - \frac{1}{d} \sum_{\chi \in \mathcal{X}} \chi((1+\xi x^d)b_j-1) \overline{\chi(\xi)} \right) = (p-1)(1-d^{-1})^k + R, \end{aligned}$$

where the first term in the last formula is just a contribution of the principal character, and the error term R absorbs the rest. It is easy to check (or see [24]) that the Weil bound implies $|R| \ll k 2^k d (1-d^{-1})^k \sqrt{p}$. Whence we obtain $|B'| = k \gg \log(\sqrt{p}/d) \gg \log p$. In view of [18] we can assume that $d \geq 3$. For $d \geq 3$ choose $k_* = \lceil \log(\sqrt{p}/d) / \log d \rceil \leq k$ elements from B . Using inclusion (19), the choice of the parameter k_* and the previous arguments, we have

$$|A| \ll p/d^{k_*} + k_* \sqrt{p} \ll dp^{1/2} \log p \cdot (\log d)^{-1} \leq p^{1/2+o(1)} \leq |\Gamma|^{1/2+o(1)}$$

(see [24]). Whence $|B| \geq |\Gamma|/|A| \geq |\Gamma|^{1/2+o(1)}$ as required. \square

Remark 18 Let $A = \{0, 1\}$, $B = (\Gamma-1) \setminus \{0\}$ or $A = (\Gamma-1) \setminus \{0\}$, $B = \{0, 1\}$. Then it is easy to check that $AB = \Gamma-1$. Thus, we need $|A \setminus \{0\}|, |B \setminus \{0\}| > 1$ in general.

Suppose that $A/B = \Gamma + x$. As in the beginning of the proof of Lemma 17 we dividing the last identity by x and redefine A such that $A/B = \xi\Gamma + 1$, where $\xi = 1/x$. In other words, for any $a \in A$, $b \in B \setminus \{0\}$ one has $\frac{a}{b} - 1 = \frac{a-b}{b} \in \xi\Gamma$. Whence for all $b, b' \neq 0$ and $a \neq b, b'$ the following holds

$$\frac{1/b - 1/a}{1/b' - 1/a} = \frac{(a-b)b'}{(a-b')b} = \frac{a-b}{b} : \frac{a-b'}{b'} \in \xi\Gamma/\xi\Gamma = \Gamma. \quad (20)$$

Similarly, if $b \neq 0$ and $a, a' \neq b$, then

$$\frac{a-b}{a'-b} = \frac{a-b}{b} : \frac{a'-b}{b} \in \xi\Gamma/\xi\Gamma = \Gamma. \quad (21)$$

In other words, $\mathsf{T}[A^{-1}, B^{-1}, B^{-1}], \mathsf{T}[B, A, A] \subseteq \Gamma \sqcup \{0\}$ and using Proposition 11 (with $A = B = A$ or $A = B = A^{-1}$), as well as the proof of Corollary 13 in the symmetric case $A = B$, we obtain that $A/A \neq \xi\Gamma + 1$ for $|\Gamma| \leq p^{2/3-\varepsilon}$. Actually, in the case $A = B$ a stronger result takes place (it is parallel to Theorem 36 from [14]).

Theorem 19 *Let $\Gamma \subset \mathbb{F}_p$ be a multiplicative subgroup, and $\xi \neq 0$ be an arbitrary residue. Suppose that for some $A \subset \mathbb{F}_p$ one has*

$$A/A \subseteq \xi\Gamma + 1. \quad (22)$$

If $|\Gamma| < p^{3/4}$, then $|A| \ll |\Gamma|^{5/12} \log^{7/6} |\Gamma|$. If $p^{3/4} \leq |\Gamma| \leq p^{5/6}$, then $|A| \ll p^{-5/8} |\Gamma|^{5/4} \log^{7/6} |\Gamma|$. If $|\Gamma| \geq p^{5/6}$, then $|A| \ll p^{-1} |\Gamma|^{5/3} \log^{1/3} |\Gamma|$.

In particular, for any $\varepsilon > 0$ and sufficiently large Γ , $|\Gamma| \leq p^{6/7-\varepsilon}$ the following holds

$$A/A \neq \xi\Gamma + 1.$$

Proof. We can assume that $|A \setminus \{0\}| > 1$. Put $\Gamma_* = \Gamma \sqcup \{0\}$ and $R = \mathsf{T}[A]$. Then in the light of (21), we have

$$R \subseteq (\xi\Gamma \sqcup \{0\})/\xi\Gamma = \Gamma_*. \quad (23)$$

Suppose that $|\Gamma| < p^{3/4}$. Applying (6) and formula (10) with $k = 1$, we obtain

$$|R| = |R \cap (1 - R)| \leq |\Gamma_* \cap (1 - \Gamma_*)| \leq |\Gamma \cap (1 - \Gamma)| + 2 \ll |\Gamma|^{2/3} \leq p^{2/3}.$$

Thus, by the third part of Theorem 7, we get $|R| \gg |A|^{8/5} / \log^{28/15} |A|$. Hence

$$\frac{|A|^{8/5}}{\log^{28/15} |A|} \ll |\Gamma|^{2/3}$$

and we obtain the required result.

Now let us suppose that $|\Gamma| \geq p^{3/4}$ but $|\Gamma| \leq p^{5/6}$. In this case, by formula (11) and the previous calculations, we get

$$|R| \ll |\Gamma|^2/p \leq p^{2/3}.$$

Applying the third part of Theorem 7 one more time, we obtain

$$\frac{|A|^{8/5}}{\log^{28/15} |A|} \ll |\Gamma|^2/p$$

and it gives the required bound for size of A , namely,

$$|A| \ll p^{-5/8} |\Gamma|^{5/4} \log^{7/6} |\Gamma|. \quad (24)$$

Now suppose that $|\Gamma| \geq p^{5/6}$. Put $q(x, y) = q_{A, A, A, A}(x, y)$. We use the second formula from (3) and note that if $q(x, y) > 0$, then $x - y \in \Gamma_*$. By inclusion (23), Theorem 4 and the Cauchy–Schwarz inequality, we obtain

$$|A|^8 \ll \left(\sum_{x, y} q(x, y) \right)^2 \leq \sum_{x, y} q^2(x, y) \cdot |\text{supp } q| \ll$$

$$\begin{aligned} &\ll \left(\frac{|A|^8}{p^2} + |A|^5 \log |A| \right) \cdot \left(\sum_{x,y} R(x)R(y)\Gamma_*(x-y) \right) \ll \\ &\ll \left(\frac{|A|^8}{p^2} + |A|^5 \log |A| \right) \cdot \left(\sum_x \Gamma_*(x)\Gamma_*(1-x) \sum_y \Gamma_*(y)\Gamma_*(1-y)\Gamma_*(x-y) \right). \end{aligned}$$

It is easy to see that the summand with $x = 1$ is negligible in the last inequality. Using formula (11) with $k = 2$ and the condition $|\Gamma| \geq p^{5/6}$, we get

$$|A|^8 \ll \left(\frac{|A|^8}{p^2} + |A|^5 \log |A| \right) \cdot \frac{|\Gamma|^5}{p^3} \ll |A|^5 \log |A| \cdot \frac{|\Gamma|^5}{p^3}.$$

It follows that

$$|A| \ll p^{-1} |\Gamma|^{5/3} \log^{1/3} |\Gamma|. \quad (25)$$

Finally, by Lemma 17, we get $|A| \sim |\Gamma|^{1/2+o(1)}$, provided $|A \setminus \{0\}| > 1$. Thus, if $|\Gamma| \leq p^{6/7-\varepsilon}$, then in view of (25) and two another bounds for size of A , we have $A/A \neq \xi\Gamma + 1$ for sufficiently large Γ . If $|A \setminus \{0\}| \leq 1$, then, clearly, $|\Gamma| \leq 2$ and this is a contradiction with the assumption $|\Gamma| \gg 1$. \square

Because our approach requires just an incidence bound from [14], [16] and Theorem 8 which are both have place in \mathbb{R} (see details in [23]), we obtain an analog of Theorem 1 as well as Theorem 19 in the real setting.

Theorem 20 *There is $\varepsilon > 0$ such that for all sufficiently large finite $A \subset \mathbb{R}$ with $|AA| \leq |A|^{1+\varepsilon}$ there is no decomposition $A + 1 = B/B$ with $|B \setminus \{0\}| > 1$.*

In a similar way, let $B \subset \mathbb{R}$ be a set such that $|B \setminus \{0\}| > 1$. Then the following holds

$$|(B/B - 1)(B/B - 1)| \gg |B/B|^{1+c},$$

where $c > 0$ is an absolute constant.

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